ME599 lecture 06 - small-scale fluid flows

As scale decreases, surface/volume increases so friction increases

→ slip counteracts this

Knudsen number = quantifies importance of slip

$$Kn = \frac{\lambda}{L_0} \quad \text{gases}$$

λ — mean free path

L_0 — length scale

$$Kn = \frac{b}{L_0} \quad \text{liquids}$$

b — slip length

e.g., L_0 = pipe diameter

start from the ideal gas law,

$$p = n k_B T$$

k_B — Boltzmann's constant, 1.38 E-23 J/K

n — # density = 2.7 E 25 /m³

mean molecular spacing, $\delta \propto n^{-1/3} \approx 3E{-9}\ m = 3\ nm$

d = hard sphere diameter $\approx 10^{-10}\ m$

so we define a "dilute gas" ⇒ $\delta/d \gg 1$

⇒ mostly atom-atom collisions

rather than multi-atom collisions

gas mean free path

$$\lambda = \frac{1}{\pi d^2 n \sqrt{2}} = \frac{kT}{\pi p d^2 \sqrt{2}}$$, substituting the ideal gas law

d = hard sphere diameter

n = f(pressure)

flow regimes:

$Kn < 0.01$ = continuum (no slip)

$0.01 < Kn < 0.1$ = slip

$0.1 < Kn < 10$ = transition

$Kn > 10$ = "free molecule" ~ macroscopic property meanings break down

⇒ direct simulations of Boltzmann equation

example of λ: air @ 1 atm, 300 K, λ = 65 nm

0.001 atm, 300 K, λ = 65 μm !

1 atm, 1200 K, λ = 2.6 μm

⇓

microchannel < 250 μm is in slip regime

Navier - stokes eqns for unsteady incompressible flow

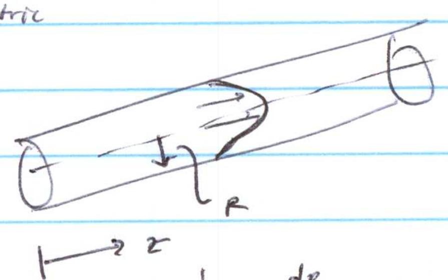
$$\frac{d\vec{v}}{dt} = \frac{-\nabla p}{\rho} + \nu^2 \nabla \vec{v}^2 + f, \quad \nu = \frac{\mu}{\rho}$$

$$\nabla \cdot \vec{v} = 0$$

$\Rightarrow$ let's quantify the effect of slip on friction in a small pipe

flow is 1-dimensional, θ-symmetric

reduced N/S eqn.



$$+\frac{1}{\mu}\frac{dp}{dz} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv(r)}{dr}\right), \quad \frac{dp}{dz} = \text{constant.}$$

steady, fully-developed flow

solve for velocity profile in the pipe, $v(r)$

tangential velocity is neglected here

$$\iint +\frac{r}{\mu}\frac{dp}{dz} = \iint \frac{d}{dr}\left(r\frac{dv(r)}{dr}\right)$$

$$\int \frac{\left(+\frac{r^2}{2\mu}\frac{dp}{dz} + c\right)}{r} = \int \frac{dv(r)}{dr}$$

$$+\frac{r^2}{4\mu}\frac{dp}{dz} + c\ln r + D = v(r) \quad \Rightarrow v(0) \text{ is finite so } c = 0$$

for no-slip, $v(R) = 0$, so $D = -\frac{R^2}{4\mu}\frac{dp}{dz}$